Lecture 22

11.2 - Series

In the last section, we talked about sequences... in this section, we add up those lists of numbers. Such a sum is called a series. Just like sequences, series can be infinite or finite. Given a sequence $\{a_n\}_{n=1}^{\infty}$, the series which is adding up its terms is denoted:

$$a_1 + a_2 + a_3 + \cdots = \sum_{j=1}^{\infty} a_j$$

Ex: $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Def: Given a series $\displaystyle\sum_{j=1}^{\infty} a_j$, the n^{th} partial sum is

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = S$, then we say the series $\displaystyle\sum_{j=1}^{\infty} a_j$ is convergent, and we let

$$\boxed{\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} s_n = S}$$

S is called the sum of the series. Otherwise, we call the series divergent.

Ex: Find a formula for the n^{th} partial sum of the series $\sum_{j=1}^{\infty} \frac{1}{2^j}$. Is this series convergent? If so, what is its sum?

Sol:
$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$,

$\ldots$, $S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$$

$\boxed{\text{Converges to 1.}}$

Ex: Repeat the last example for the series $\sum_{j=1}^{\infty} j$.

Sol: Recall $\sum_{j=1}^{n} j = \frac{n(n+1)}{2} = S_n$

$$\sum_{j=1}^{\infty} j = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty.$$

$\boxed{\text{Divergent}}$

Geometric Series

A <u>geometric series</u> is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

<u>Theorem</u>: The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r|<1$, and it converges to $\frac{a}{1-r}$.

If $|r| \geq 1$, the series diverges.

<u>proof</u>: If $r=1$, $\sum_{n=1}^{\infty} ar^{n-1} = a+a+a+\cdots = \infty$, divergent.

If $r=-1$, $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} a$

The partial sums are $S_m = \begin{cases} a, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$

$\lim_{m\to\infty} S_m$ does not exist, so the series diverges.

If $|r| \neq 1$, $S_m = a + ar + \cdots + ar^{m-1}$

$rS_m = ar + \cdots + ar^{m-1} + ar^m$

$\Rightarrow (1-r)S_m = S_m - rS_m = a - ar^m \Rightarrow S_m = \frac{a}{1-r} - \frac{ar^m}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{m\to\infty}\left(\frac{a}{1-r} - \frac{ar^m}{1-r}\right) = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ -\infty, & |r|>1 \end{cases}$$

Ex: Find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^n}$$

Sol:

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^n} = \sum_{n=1}^{\infty} 3\left(\frac{-1}{5}\right)^n = \sum_{n=1}^{\infty} \frac{-3}{5}\left(\frac{-1}{5}\right)^{n-1}$$

$a = \frac{-3}{5}$

$r = \frac{-1}{5}$

$$= \frac{-3/5}{1-(-1/5)} = \frac{-3/5}{6/5} = \boxed{\frac{-1}{2}}$$

Ex: Express $0.777777\cdots = 0.\overline{7}$ as a fraction.

Sol: $0.7777\cdots = 0.7 + 0.07 + 0.007 + 0.0007 + \cdots$

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{7}{10^n} = \sum_{n=1}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^{n-1}$$

$$= \frac{7/10}{1-1/10} = \frac{7/10}{9/10} = \boxed{\frac{7}{9}}$$

Telescoping Series

A telescoping series is one of the form:

$$\sum_{j=1}^{\infty}\left(f(j)-f(j+1)\right)=\left(f(1)-f(2)\right)+\left(f(2)-f(3)\right)+\left(f(3)-f(4)\right)+\cdots$$

Because of its form, its partial sums are pretty simple: $S_n = f(1)-f(n+1)$

Ex: Determine the sum of:

$$\sum_{j=1}^{\infty}\frac{1}{j^2+3j+2}$$

Sol: $\displaystyle\sum_{j=1}^{\infty}\frac{1}{j^2+3j+2}=\sum_{j=1}^{\infty}\left(\frac{1}{j+1}-\frac{1}{j+2}\right)$ $\qquad f(j)=\dfrac{1}{j+1}$

$f(j+1)=\dfrac{1}{j+2}$

$$S_n = f(1)-f(n+1)=\frac{1}{2}-\frac{1}{n+2}$$

$$\sum_{j=1}^{\infty}\frac{1}{j^2+3j+2}=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\boxed{\frac{1}{2}}$$

Harmonic Series

The harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, is an example of a divergent series. This example is important for comparison tests and counter examples. We'll prove it diverges in the next section.

Remark: Adding or removing a finite # of terms from the beginning of a series will not affect its convergence (or divergence), e.g.,

$\sum_{j=-4}^{\infty} \frac{1}{2^j}$ is still convergent & $\sum_{j=1000}^{\infty} \frac{1}{j}$ is still divergent.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

⚠ The converse is NOT TRUE! That is:

$\lim_{n\to\infty} a_n = 0$ does not imply that $\sum_{n=1}^{\infty} a_n$ converges, e.g. the harmonic series.

Ex: Do the series:

(a) $\sum_{n=1}^{\infty} \frac{n^3}{2n^3-2}$ & (b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

diverge?

Sol: (a) $\lim_{n\to\infty} \frac{n^3}{2n^3-2} = \frac{1}{2}$, so (a) diverges

(b) $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$, so the test is inconclusive.

Properties of Series

Suppose Σa_n & Σb_n are convergent and c is a number. Then $\Sigma(a_n+b_n)$, $\Sigma(a_n-b_n)$, and Σca_n all converge, and

$$\Sigma(a_n+b_n) = \Sigma a_n + \Sigma b_n \qquad \Sigma(a_n-b_n) = \Sigma a_n - \Sigma b_n$$

$$\Sigma ca_n = c\Sigma a_n$$

Ex: Find the sum of

$$\sum_{n=1}^{\infty}\left(\frac{1+2^n}{3^n}\right)$$

Sol: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+2^n}{3^n}\right) = \sum_{n=1}^{\infty}\frac{1}{3^n} + \sum_{n=1}^{\infty}\frac{2^n}{3^n} = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$

$$= \frac{1/3}{1-1/3} + \frac{2/3}{1-2/3} = \frac{1/3}{2/3} + \frac{2/3}{1/3} = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$$

Ex: For what values of x does $\displaystyle\sum_{n=1}^{\infty}\frac{2x^n}{7^{n-1}}$ converge?

Sol: $\displaystyle\sum_{n=1}^{\infty}\frac{2x^n}{7^{n-1}} = \sum_{n=1}^{\infty}2x\left(\frac{x}{7}\right)^{n-1}$. Converges if $\left|\frac{x}{7}\right| < 1$

$\Rightarrow -1 < \frac{x}{7} < 1 \Rightarrow \boxed{-7 < x < 7}$

Ex: Does $\displaystyle\sum_{n=10}^{\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverge?

Sol: $\displaystyle\lim_{n\to\infty}\ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0$

The series diverges.